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LETTER TO THE EDITOR

The boson realizations of the quantum group $U_q(sl(2))$

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Abstract. The method of construction of boson realizations of semisimple Lie algebras formulated in our previous paper ia applied to the case of the quantum group $U_q(sl(2))$. The realizations are expressed in terms of the usual Weyl algebra.

The theory of quantum groups has attracted great interest recently. In mathematics quantum groups are related to Hopf algebra (Drinfeld 1986) non-commutative geometry (Podles 1987) and the theory of knots and links (Witten 1989). In physics they are relevant for the theory of integrable systems (Sklyanin 1982, Kulish and Reshitikin 1981), certain problems in statistical physics and the study of conformal field theories in two dimensions (Smit 1990).

A realization (also canonical realization or boson representation) of a Lie algebra g denotes an expression of elements of g by means of polynomials in quantum variables a_i, a_i^+, which preserves the commutation relations of g.

The construction of boson realizations of semisimple Lie algebras represents an interesting problem both mathematically and physically (Barut and Raczka 1977). In physics boson realizations of Lie algebras are used to study physical systems with symmetries in the framework of the canonical formalism (Schwinger 1965).

In the paper by Burdík (1985) the method of constructing realizations for an arbitrary real semisimple algebra g was presented. It was shown that any induced representation can be rewritten as the so-called boson representation. The construction starts from a triangle decomposition $g = n_+ \oplus h \oplus n_-$ (Dixmier 1974).

In recent papers (Biedenharn 1989, Macfarlane 1989, Hyashi 1990) the authors gave the q-oscillator representations of the quantum groups using a one-parameter quantum deformation of the Weyl algebra. But if we want to give a representation of the quantum groups by usual means we must study the representations of a deformed Weyl algebra.

The purpose of this letter is to use our method of constructing realizations to the quantum case of algebra $U_q(sl(2))$. We obtain the realization in terms of one canonical pair of the Weyl algebra W_2. We believe that our method can be used to construct other simple quantum groups.

Definition. The Weyl algebra W_2 is the associative algebra over C with the identity generated by two elements a and a^+ which satisfy the relations

$$[a, a^+] = 1. \tag{1}$$

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Because we have:

$$[a,(aa^+)^k]=\sum_{l=1}^{k}\binom{k}{l}(aa^+)^{k-l}a \tag{2}$$

$$[a^+,(aa^+)^k]=\sum_{l=1}^{k}\binom{k}{l}(-1)^l(aa^+)^{k-l}a^+$$

we can complete W_2 by the analytic functions of aa^+.

We will use the functions:

$$e^{\beta aa^+}=\sum_{k=0}^{\infty}\frac{(\beta aa^+)^k}{k!}\qquad f_1(\beta aa^+)=\sum_{k=0}^{\infty}\frac{(\beta aa^+)^k}{(k+1)!} \tag{3}$$

and we obtain using the above relations (2) that

$$\begin{aligned}
&[a, a^+f_1(\beta aa^+)]\\
&=aa^+f_1(\beta aa^+)-\sum_{k=0}^{\infty}a^+\frac{(\beta aa^+)^k}{(k+1)!}a\\
&=aa^+f_1(\beta aa^+)-\sum_{k=0}^{\infty}\sum_{l=0}^{k}\frac{\beta^k}{(k+1)!}\binom{k}{l}(-1)^l(aa^+)^{k-1}a^+a\\
&=aa^+f_1(\beta aa^+)+\sum_{k=0}^{\infty}\sum_{l=0}^{k}\frac{\beta^k}{(k+1)!}\binom{k}{l}(-1)^l(aa^+)^{k-l}\\
&\quad-\sum_{k=0}^{\infty}\sum_{l=0}^{k}\frac{\beta^k}{(k+1)!}\binom{k}{l}(-1)^l(aa^+)^{k-l+1}\\
&=\sum_{k=0}^{\infty}\sum_{l=0}^{k}\frac{\beta^k}{(k+1)!}(-1)\left[\binom{k}{l+1}+\binom{k}{l}\right](aa^+)^{k-l}\\
&=\left(\sum_{k=0}^{\infty}\frac{(\beta aa^+)^k}{k!}\right)\left(\sum_{l=0}^{\infty}\frac{(-\beta)^l}{(l+1)!}\right).
\end{aligned} \tag{4}$$

Definition. The quantum group $U_q(sl(2))$ is generated algebraically by the operators J_+, J_- and J_z obeying the Lie bracket (commutator relations)

$$[J_z,J_+]=J_+ \tag{5}$$

$$[J_z,J_-]=-J_- \tag{6}$$

$$[J_+,J_-]=[J_z]_q\equiv\frac{q^{J_z}-q^{-J_z}}{q^{1/2}-q^{1/2}}. \tag{7}$$

We denote $w=\log(q)$, $\alpha=(q^{1/2}-q^{-1/2})^{-1}$ and by a very simple calculation we obtain

$$e^{wJ_z}(J_+)^n=(J_+)^n\,e^{w(J_z+n)}\qquad e^{-wJ_z}(J_+)^n=(J_+)^n\,e^{-w(J_z+n)}. \tag{8}$$

Definition. A realization of a quantum algebra $U_q(sl(2))$ is a homomorphism τ

$$\tau:U_q(sl(2))\to W_2. \tag{9}$$

In particular these relations must be accepted:

$$\begin{aligned}
&\tau([J_z,J_+])=[\tau(J_z),\tau(J_+)]=\tau(J_+)\\
&\tau([J_z,J_-])=[\tau(J_z),\tau(J_-)]=-\tau(J_-)\\
&\tau([J_+,J_-])=[\tau(J_+),\tau(J_-)]=\tau([J_z]_q).
\end{aligned} \tag{10}$$

On the other hand if we have defined $\tau(J_\pm)$, $\tau(J_z)$ and the relations (10) hold we can extend the mapping τ to the homomorphic mapping of the quantum group $U_q(\mathrm{sl}(2))$ into W_2.

The starting point of our construction for the simple Lie algebras was the triangle decomposition and Verma modules (for details see Burdík 1985). Analogously we construct the Verma modules for the quantum groups $U_q(\mathrm{sl}(2))$.

The representation space form the vectors $|n\rangle = (J_+)^n$ and for the Verma modules ρ_λ we obtain

$$\rho_\lambda(J_+)|n\rangle = J_+(J_+)^n = |n+1\rangle$$

$$\rho_\lambda(J_z)|n\rangle = J_z(J_+)^n = (J_+)^n J_z + n(J_+)^n = (n+\lambda)|n\rangle$$

$$\begin{aligned}\rho_\lambda(J_-)|n\rangle &= J_-(J_+)^n = (J_+)^n J_- - \alpha \sum_{k=0}^{n-1} (J_+)^{n-(k+1)}[J_-, J_+](J_+)^k \\ &= -\alpha(J_+)^{n-1}\left(e^{wJ_z}\sum_{k=0}^{n-1} e^{wk} - e^{-wJ_z}\sum_{k=0}^{n-1} e^{-wk}\right) \\ &= -\alpha(J_+)^{n-1}\left(e^{wJ_z}\frac{e^{wn}-1}{e^{w}-1} - e^{-wJ_z}\frac{e^{-wn}-1}{e^{-w}-1}\right) \\ &= -w\alpha^2 n (J_+)^{n-1}(e^{w(J_z-1/2)} f_1(wn) - e^{-w(J_z-1/2)} f_1(-wn)) \\ &= -w\alpha^2 (e^{w(\lambda-1/2)} f_1(wn) - e^{-w(\lambda-1/2)} f_1(-wn)) n|n-1\rangle.\end{aligned} \tag{11}$$

Now we obtain the realization of the quantum algebra $U_q(\mathrm{sl}(2))$ in this way. If we define on the representation space the operators a, a^+ by the relations

$$\begin{aligned} a|n\rangle &= |n+1\rangle \\ a^+|n\rangle &= -n|n-1\rangle \end{aligned} \tag{12}$$

we can rewrite the Verma modules ρ_λ using the operators

$$\begin{aligned}\rho_\lambda(J_+) &= a \\ \rho_\lambda(J_z) &= -aa^+ + \lambda \\ \rho_\lambda(J_-) &= w\alpha^2 a^+(e^{w(\lambda-1/2)} f_1(-waa^+) - e^{-w(\lambda l-1/2)} f_1(waa^+)).\end{aligned} \tag{13}$$

By direct calculation we prove that the mapping (13) fulfils the relations (10) abstractly and not only on ket vectors that terminate with the vacuum ket.

In this letter we have presented a construction of boson realizations of the quantum group $U_q(\mathrm{sl}(2))$. The realizations are in the usual quantum variables a, a^+ without deformations used in previous papers (Biedenharn 1989, Macfarlane 1989, Hyashi 1990) and are consequently the realization on the abstract level not only in the vacuum representation (see a remark in Bidenharn 1989 p L875). We premise that a realization of this type could play a role in representation theory and the application of the quantum group in physics and mathematics because the representation theory of the usual Weyl group is very well known. However, we expect that our method gives the realization for other quantum groups. Some positive indications have been obtained already and the results will be presented in a forthcoming paper.

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